siderably more reduced when fluid is injected into the fixed tube then it is in the case of suction.

Existence of Points of Inflexion

The existence of points of inflection in the velocity profile is of importance in flow stability considerations.[2] They are given by those values of η which satisfy the relations

$$d^2V_x/d\eta^2 = 0; \quad d^3V_x/d\eta^3 \neq 0 \tag{17}$$

Case 1. Injection Reynolds Number

Using Eq. (10) for V_x, the point of inflection in this case is given by

$$\eta = \left[\frac{2c(1-\lambda^R)}{R(R-1)\{c(1-\lambda^2) - 2(R-2)\}}\right]^{1/(R-2)} \tag{18}$$

The condition that it should lie in the flow region—i.e., $\lambda \leq \eta \leq 1$—for $R > 2$ is that

$$\frac{2R(R-1)(R-2)\lambda^{R-2}}{2(1-\lambda^R) - R(R-1)(\lambda^{R-2}-\lambda^R)} \leq \frac{Re}{2}\frac{\partial\varpi}{\partial\xi} \leq \frac{2R(R-1)(R-2)}{2(1-\lambda^R) - R(R-1)(1-\lambda^2)} \tag{19}$$

For injection Reynolds numbers less than 2, the signs of inequality are altered in Eq. (19).

Case 2. Suction Reynolds Number

Using Eq. (11), the point of inflection in the velocity profile is given in this case by

$$\eta = \left[\frac{R(R+1)\{c\lambda^R(\lambda^2-1) - 2(R+2)\lambda^R\}}{2c(1-\lambda^R)}\right]^{1/(R+2)} \tag{20}$$

and the limits on the pressure gradient so that η [given by Eq. (20)] should lie in the flow region are given by the inequality

$$\frac{2R(R+1)(R+2)}{2\lambda^2(1-\lambda^R) + R(R+1)(1-\lambda^2)} \geq \frac{Re}{2}\frac{\partial\varpi}{\partial\xi} \geq \frac{2R(R+1)(R+2)\lambda^R}{2(1-\lambda^R) + R(R+1)(1-\lambda^2)\lambda^R} \tag{21}$$

Special Case—Injection Reynolds Number $Re = 2$

In this case, the solution of Eq. (6) subject to the boundary conditions (9) is

$$V_x = \left(\frac{c}{2}\ln\lambda - 1\right)\frac{\lambda^2}{1-\lambda^2} + \frac{[1-(c/2)\lambda^2\ln\lambda]}{1-\lambda^2}\eta^2 - \frac{c}{2}\eta^2\ln\eta \tag{22}$$

The value of c for which separation occurs at the fixed tube is

$$c = 4/(1-\lambda^2+2\ln\lambda) \tag{23}$$

whence

$$\partial\varpi/\partial\xi = -8/[Re(1-\lambda^2+2\ln\lambda)] \tag{24}$$

The skin friction at the fixed tube is given by

$$\tau/(\rho U^2) = (1/Re)[2\lambda/(1-\lambda^2)] \tag{25}$$

Fig. 1 shows that for a fixed Re, the axial pressure gradient which would provoke separation at the fixed tube increases asymptotically as $\lambda \to 1$ for both suction and injection, and also that it is less in the case of injection than that of suction. Fig. 2 shows that skin friction (for a fixed Re) is less in the case of injection than that of suction and also that for injection it increases asymptotically as $\lambda \to 1$, and for suction it increases asymptotically for annulus radius ratio tending to both 0 and 1.

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The Characteristic Form of the Equations of One-Dimensional Magnetohydrodynamic Flow With Oblique Magnetic Field

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This note presents a derivation of the characteristic form of the equations governing the one-dimensional unsteady flow of an ideal, inviscid, perfectly conducting, compressible fluid subjected to an oblique magnetic field. It is assumed that all dependent variables are functions of only one space variable x and the time t, i.e., it is the projections of the characteristics (really, bicharacteristics) from (x,y,t) space onto the (x,t) plane that are considered. The resultant characteristic system, which contains the characteristic systems for a purely transverse field[4–6] and the nonmagnetic case[1] as special cases, is suitable for approximate treatment by finite-difference techniques.

The basic equations are:[3]

$$c_t + uc_x + (\gamma-1)cu_x/2 = 0 \tag{1}$$

$$u_t + uu_x + 2cc_x/(\gamma-1) + b_2^2B_{2x}/B_2 - c^2s_x/\gamma(\gamma-1)c_v = 0 \tag{2}$$

$$v_t + uv_x - b_1^2B_{2x}/B_1 = 0 \tag{3}$$

$$B_{2t} - B_1v_x + u_xB_2 + uB_{2x} = 0 \tag{4}$$

$$s_t + us_x = 0 \tag{5}$$

where $\mathbf{q} = (u,v,0)$, $\mathbf{B} = (B_1, B_2, 0)$, c, s, $b_i^2 = B_i^2/\mu\rho (i = 1, 2)$, ρ, μ and γ are, respectively, the particle velocity, induction, local speed of sound, specific entropy, square of the Alfvén speed, density, permeability, and ratio of specific heats at constant pressure c_p and at constant volume c_v. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone. As a consequence of Maxwell's equations, there is the further condition that B_1 be constant.

To determine the characteristic curves of the system (1)–(5), it is convenient to introduce[2] new independent variables $\phi(x,t)$, $\psi(x,t)$. The possible characteristic manifolds are given by solutions of the first-order partial differential equation for ϕ:

$$[\phi_t + u\phi_x][\phi_t^4 + 4u\phi_x\phi_t^3 + (6u^2 - b_1^2 - b_2^2 - c^2)\phi_x^2\phi_t^2 + \{4u^3 - 2u(b_1^2+b_2^2+c^2)\}\phi_x^3\phi_t + \phi_x^4\{u^2(u^2-b_1^2-b_2^2) - c^2(u^2-b_1^2)\}] = 0 \tag{6}$$

The first factor shows that one characteristic curve is given by $dx/dt = u$. If $\phi(x,t) =$ constant is characteristic, then it follows that $dx/dt = -\phi_t/\phi_x$, so that the introduction of $\lambda = dx/dt$ in the remaining factor of (6) leads to a fourth-order algebraic equation for λ. This takes a better form if the substitution $\lambda = u + a$ is made; the resultant algebraic equation for a is

$$a^4 - \omega^2a^2 + b_1^2c^2 = 0 \tag{7}$$

where $\omega^2 = c^2 + b_1^2 + b_2^2$. The larger and the smaller of the roots $a > 0$ of (7) will be denoted by a_f (fast speed) and a_s (slow speed), respectively.[3]

Thus, introducing characteristic parameters $(\alpha, \beta, \xi, \eta, \zeta)$, the first five equations of the characteristic system are

$$\left.\begin{aligned} x_\beta = (u+a_f)t_\beta,\ x_\alpha = (u-a_f)t_\alpha,\ x_\xi = ut_\xi \\ x_\eta = (u+a_s)t_\eta,\ x_\zeta = (u-a_s)t_\zeta \end{aligned}\right\} \tag{8}$$

To determine the remaining equations of the characteristic system, let the equations (1)–(5) be multiplied by ν_1, ν_2, ν_5, ν_3, ν_4, respectively, and add the resultant equations:

$$\left.\begin{aligned} &[\nu_1(\gamma-1)c/2 + \nu_2u + \nu_3B_2]u_x + \nu_2u_t + \\ &[\nu_1u + 2c\nu_2/(\gamma-1)]c_x + \nu_1c_t + [\nu_5u - \nu_3B_1]v_x + \\ &\nu_5v_t + [\nu_2b_2^2/B_2 - \nu_5b_1^2/B_1 + \nu_3u]B_x + \\ &\nu_3B_t + [\nu_4u - \nu_2c^2/\gamma(\gamma-1)c_v] + \nu_4s_t = 0 \end{aligned}\right\} \tag{9}$$

The condition that (9) contain only derivatives in the characteristic direction is expressed by equality of the ratios

$$\left.\begin{aligned}&(\nu_1 u + 2\nu_2 c/(\gamma - 1))/\nu_1 = (\nu_1(\gamma - 1)c/2 + \nu_2 u + \nu_3 B_2)/\nu_2 = \\ &(\nu_5 u - \nu_3 B_1)/\nu_5 = (\nu_2 b_2^2/B_2 - \nu_5 b_1^2/B_1 + \nu_3 u)/\nu_3 = \\ &(\nu_4 u - \nu_2 c^2/\gamma(\gamma - 1)c_v)/\nu_4 = dx/dt = u + a\end{aligned}\right\} \quad (10)$$

The solutions for ν_j/ν_1 ($j = 2, 3, 4, 5$) are

$$\left.\begin{aligned}&\frac{\nu_2}{\nu_1} = \frac{(\gamma - 1)a}{2c}, \quad \frac{\nu_3}{\nu_1} = \frac{(\gamma - 1)b_2^2 a^2}{2cB_2(a^2 - b_1^2)} \\ &\frac{\nu_4}{\nu_1} = -\frac{c}{2\gamma c_v}, \quad \frac{\nu_5}{\nu_1} = \frac{-(\gamma - 1)B_1 b_2^2 a}{2cB_2(a^2 - b_1^2)}\end{aligned}\right\} \quad (11)$$

The remaining five equations of the characteristic system are obtained by substituting (11) into (9) for the three cases of interest, viz., $a = a_f, a_s, 0$. This gives

$$a_f u_\beta + 2cc_\beta/(\gamma - 1) + (a_f^2 - c^2)B_{2\beta}/B_2 - B_1(a_f^2 - c^2)v_\beta/a_f B_2 - c^2 s_\beta/\gamma(\gamma - 1)c_v = 0 \quad (12)$$

$$-a_f u_\alpha + 2cc_\alpha/(\gamma - 1) + (a_f^2 - c^2)B_{2\alpha}/B_2 + B_1(a_f^2 - c^2)v_\alpha/a_f B_2 - c^2 s_\alpha/\gamma(\gamma - 1)c_v = 0 \quad (13)$$

$$a_s u_\eta + 2cc_\eta/(\gamma - 1) + (a_s^2 - c^2)B_{2\eta}/B_2 - B_1(a_s^2 - c^2)v_\eta/a_f B_2 - c^2 s_\eta/\gamma(\gamma - 1)c_v = 0 \quad (14)$$

$$-a_s u_\zeta + 2cc_\zeta/(\gamma - 1) + (a_s^2 - c^2)B_{2\zeta}/B_2 + B_1(a_s^2 - c^2)v_\zeta/a_s B_2 - c^2 s_\zeta/\gamma(\gamma - 1)c_v = 0 \quad (15)$$

$$s_\zeta = 0 \quad (16)$$

The characteristic form of the system (1)–(5) consists of (8) and (12)–(16). For the case of a transverse field only, i.e., $B_1 = 0$, $a_s = 0$ and $a_f = \omega = [b_2^2 + c^2]^{1/2}$, the characteristic system reduces exactly to that previously presented.[4–6]

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The Free-Molecule Impact-Pressure Probe of Arbitrary Length†

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THIS NOTE is written to supplement the work recently published by Pond,[1] who calculated the probabilities of molecules passing through cylindrical tubes for selected values of entrance velocity and tube length. The results reported here agree with those of Pond's where comparison is possible, but were obtained by a completely independent and different analysis of the underlying theory late in 1961. Our analysis of the free-molecule impact-pressure probe was based on a direct numerical integration of the relevant equations set up by Harris and Patterson.[2] For completeness these are stated below. The calculations were performed with the aid of an IBM-650 computer for specific speed ratios between 0 and 20 and for tube geometries ranging continuously from an orifice to an infinitely long tube.

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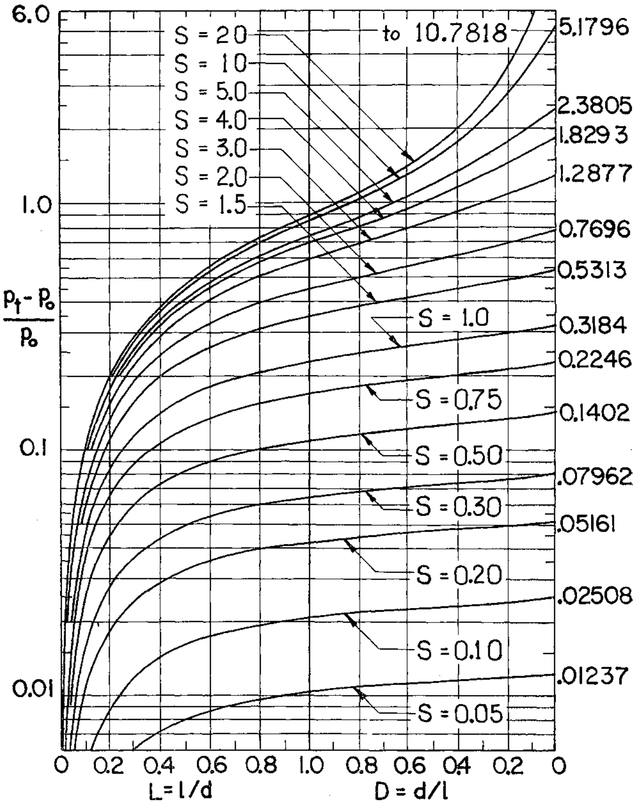

FIG. 1. Fractional deviation of impact-tube pressure from orifice gauge pressure, $(p_t - p_0)/p_0$, as a function of tube geometry.

The gage volume pressure p_t behind an impact tube may be related to the free-stream static pressure p_1 by means of probability functions W, which depend on the speed ratio S of the flow and the diameter-to-length ratio D of the tube:

$$p_t\sqrt{T_1}/p_1\sqrt{T_g} = W(S,D)/W(0,D) \quad (1)$$

In this expression T_1 and T_g are the free-stream and gage temperatures, respectively, and

$$\left.\begin{aligned}
W(S,D) &= \alpha\left[2\chi(S) - e^{-S^2}\psi(D) - \frac{4S}{\sqrt{\pi}}\eta(S,D)\right] + (1 - 2\alpha)\left[\chi(S) - e^{-S^2}\zeta(D) - \frac{4S}{\sqrt{\pi}}\mu(S,D)\right] \\
\chi(S) &= e^{-S^2} + S\sqrt{\pi}(1 + \operatorname{erf} S) \\
\psi(D) &= 2(\sqrt{1 + D^2} - 1)/D^2 \\
\zeta(D) &= 2[(1 + D^2)^{3/2} - D^3 - 1]/3D^2 \\
\eta(S,D) &= \frac{1}{D}\int_0^1 dY \times \int_0^{\sqrt{(1-Y^2)/(1-Y^2+1/D^2)}} [1 + \operatorname{erf}(S\sqrt{1 - t^2})]e^{-S^2t^2}dt \\
\mu(S,D) &= \int_0^1 \eta(S,D/X)dX
\end{aligned}\right\} \quad (2)$$

The quantity $\alpha(D)$ is the Clausing[4] probability function appearing in

$$w(X,D) = \alpha + (1 - 2\alpha)X \quad (3)$$